On the Location of the 1-particle Branch of the Spectrum of the Disordered Stochastic Ising Model

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Abstract

We analyse the lower non trivial part of the spectrum of the generator of the Glauber dynamics for a d-dimensional nearest neighbour Ising model with a bounded random potential. We prove conjecture 1 in [AMSZ]: for sufficently large values of the temperature, the first band of the spectrum of the generator of the process coincides with a closed non random segment of the real line.

1 Introduction

In [AMSZ] the authors study the generator of the Glauber dynamics for a 1-d Ising model with random bounded potential. They prove that, for any realization of the potential and any value of the inverse temperature $\beta > 0$, the spectrum of the generator is the union of disjoint closed subsets of the real line (k- particles branches, $k \in \mathbb{N}^+$) and, with probability one with respect to the distribution of the potential, is a non random set. In particular it is proved there that there exists a spectral gap and thus the model exhibits exponential relaxation to

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equilibrium. As it is to be expected, and proved in [AMSZ], a relaxation rate which is valid for every realization is the same as for the non-disordered model with a coupling constant that coincides with the maximum value of the coupling in the disordered model. For the average over the disorder of the single spin autocorrelation function, the speed of relaxation is somewhat larger as proved in [Zh].

Boundedness of the potential is essential for all these results of fast convergence to equilibrium. In this case a fairly detailed information on parts of the spectrum of the generator is available ([AMSZ], [Zh]). Also in more than 1-d convergence faster than exponential in the average can be proved at high temperature [CMM].

When the interactions are not bounded the situation is considerably different. Even in 1-d there is no spectral gap (see [Ze]) and relaxation rate is subexponential (see [SZ]).

In [AMSZ] it is conjectured that (conjecture 1, page 657) that results similar to those proved there for 1-d should hold for β small enough in dimensions $d \geq 2$. It can be readily seen that for the proof, in 1-d, of the results conjectured to be true in $d \geq 2$, the assumption of ferromagnetic coupling is not needed. It is only used later to prove exponential decay of eigenfunctions.

In this work we consider the Glauber dynamics for d-dimensional nearest neighbour Ising model, with bounded random potential having absolutely continuous distribution with respect to the Lebesgue measure and prove that conjecture 1 in [AMSZ] is true.

That is, there exists a constant C, depending on the distribution of the potential and on the lattice dimension d, such that, at high temperature, the first branch of the spectrum of the generator of the process, at first order in β, coincides, for almost every realization of the potential, with the segment

$$[1 - C\beta, 1 + C\beta]$$

(for a more precise statement see Theorem 3). In particular this implies that, at first order in β, the spectral gap is larger than $1 - C\beta$.

We remark that at lower temperatures, but still in the uniqueness region, relaxation is strictly slower than exponential for almost every realization of the potential (see Theorem 3.3 of [CMM]).

2 Notations and Results

Consider the lattice $\mathbb{Z}^d$ and the set of bonds of the lattice $\mathbb{B}_d := \left\{ \{x, y\} \subset \mathbb{Z}^d : |x - y| = 1 \right\}$. We introduce a collection of i.i.d random variables indexed by $\mathbb{B}_d$. On each bond of the lattice we define a random variable

$$\omega_b \in \left[J^-, J^+ \right], \ \mathbb{B}_d \ni b,$$

whose probability distribution is absolutely continuous with respect to the Lebesgue measure. The random field ω is a function on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\Omega = \left[J^-, J^+\right]^{\mathbb{B}_d}$, and is ergodic w.r.t. the the group of automorphisms on Ω generated by the lattice shift.

We now consider an Ising system in $\mathbb{Z}^d$, denoting by σ the spin variables and by $\mathcal{S}$ the configuration space $\{-1,+1\}^{\mathbb{Z}^d}$.

$\{\tau_z\}_{z\in\mathbb{Z}^d}$ is the group of automorphisms of $\mathcal{S}$, generated by the lattice translations

$$(\tau_z\sigma)_x = \sigma_{x-z} \quad x,z\in\mathbb{Z}^d$$

and j is the involution of $\mathcal{S}$ given by

$$\mathcal{S}\ni\sigma\longmapsto j(\sigma) = -\sigma\in\mathcal{S}\,.$$

The Hamiltonian of the models studied throughout this paper is

$$H_\Lambda^\omega(\eta|\xi_{\partial\Lambda}) = -\sum_{\Lambda\ni x,y\,:\,|x-y|=1}\frac{1}{2}\eta_x\omega_{x,y}\eta_y + \sum_{x\in\Lambda,\,y\in\Lambda^c\,:\,|x-y|=1}\eta_x\omega_{x,y}\xi_y\,, \tag{1}$$

where $\xi_{\partial\Lambda} := (\xi_i)_{i\in\partial\Lambda}$ is the boundary condition.

For any $\beta>0$ and any realization ω of the potential, let $\mathcal{G}(\beta,\omega)$ be the set of Gibbs states of the system specified by

$$\begin{aligned}\mu_\Lambda^{\beta,\omega}(d\sigma|\sigma_{\partial\Lambda}) &:= \frac{e^{-\beta H_\Lambda^\omega(\sigma|\sigma_{\partial\Lambda})}}{Z_\Lambda^{(d)}(\beta,\omega|\sigma_{\partial\Lambda})}\mu_\Lambda(d\sigma) \quad \Lambda\subset\subset\mathbb{Z}^d\\ Z_\Lambda^{(d)}(\beta,\omega|\sigma_{\partial\Lambda}) &:= \mu_\Lambda\left(e^{-\beta H_\Lambda^\omega(\sigma|\sigma_{\partial\Lambda})}\right)\,.\end{aligned}$$

$\mathcal{G}(\beta,\omega)$ is also the set of probability measures reversibles with respect to the process generated by

$$L(\beta,\omega)f := \sum_{x\in\mathbb{Z}^d} w_x^{\beta,\omega}(\sigma)\left[f(\sigma)-f(\sigma^x)\right]\,, \tag{2}$$

where σ^x represents the configuration in $\mathcal{S}$ such that

$$\sigma_y^x = \begin{cases}\sigma_y & y\neq x\\ -\sigma_y & y=x\end{cases} \qquad y\in\mathbb{Z}^d$$

and f is a cylindrical function in $L^2\left(\mathcal{S},\mu^{\beta,\omega}\right) := \mathcal{L}(\beta,\omega)$.

We will always consider the generator L a positive operator, so that $S(t)=\exp[-tL]$ will represent the associated semigroup.

In the following, with a little abuse of notation, we will use the same notation for the operator (2) and for its closure in $\mathcal{L}(\beta,\omega)$ which, by reversibility of the Gibbs measure, is also selfadjoint on $\mathcal{L}(\beta,\omega)$.

Let us define $J := |J^-|\vee|J^+|$ and $\forall x\in\mathbb{Z}^d$, $\omega\in\Omega$

$$\Delta_x H^\omega(\eta) := H^\omega(\eta) - H^\omega(\eta^x) = -2\sum_{b\ni x}\omega_b\sigma_b(\eta) = -\eta_x\sum_{y:\,|x-y|=1}\omega_{x,y}\eta_y\,, \tag{3}$$

then

$$-4dJ \leq |\Delta_x H^\omega(\sigma)| \leq 4dJ\,. \tag{4}$$

In the following we will restrict ourselves to the choice of transition rates of the form

$$w_x^{\beta,\omega}(\sigma) = \psi(\beta\Delta_x H^\omega(\sigma))\ , \tag{5}$$

where ψ is a monotone function, so that

$$\psi(-\beta 4dJ) \wedge \psi(\beta 4dJ) \leq w_x^{\beta,\omega}(\sigma) \leq \psi(-\beta 4dJ) \vee \psi(\beta 4dJ)\ . \tag{6}$$

In particular, we will work out the details for the case of the *heat bath dynamics* as was done in [AMSZ]

$$w_{hb,x}^{\beta,\omega}(\sigma) = \psi_{hb}(\beta\Delta_x H^\omega(\sigma)) = \frac{1}{1+e^{-\beta\Delta_x H^\omega(\sigma)}}. \tag{7}$$

Our analysis can be applied to any Glauber process with transition rates of the kind given in (5).

The results contained in this paper are:

Theorem 1 *There exists a value $\beta_d^{-1}(J)$ of the temperature such that, for any $\beta \in [0, \beta_d(J))$ and any realization of the potential ω, the first non trivial branch of the spectrum of the generator of the heat bath dynamics, $\sigma_\beta^{(1)}$, is contained in the interval $\left[g_d^-(\beta), g_d^+(\beta)\right]$ where $g_d^-(\beta), g_d^+(\beta)$ are analytic functions of β such that*

$$g_d^\pm(\beta) = 1 \pm 2dJ\beta + o(\beta)\,.$$

For a definition of $\sigma_\beta^{(1)}$ and a discussion of its relevance see Corollary 1 of [AMSZ] and Theorem 2.3 of [M].

Theorem 2 *There exists a value $\beta_d^{(1)}$ of β such that, for almost every $\beta \in \left[0, \beta_d^{(1)}\right)$ and any realization of the potential ω, the first non trivial branch of the spectrum of the generator $\sigma_\beta^{(1)}$ satisfies*

$$\left[1 - f_d^-(\beta), 1 + f_d^+(\beta)\right] \subseteq \sigma_\beta^{(1)}\ ,$$

where $f_d^-(\beta)$, $f_d^+(\beta)$ are analytic functions of β such that

$$f_d^\pm(\beta) = \pm 2dJ\beta + o(\beta)\,.$$

Remark 1. The analiticity of the functions introduced in the above two theorems, does not hold only for the heat bath dynamics, but is guaranteed for any dynamics where ψ is an analytic function. If this is not the case, the statement about analyticity must be dropped from the above theorems.

Theorem 3 *There exists a value $\beta_d^*(J) \leq \beta_d^{(1)} \wedge \beta_d(J)$ of β such that, for every $\beta \in [0, \beta_d^*(J))$ and any realization of the potential ω, the first non trivial branch of the spectrum of the generator of the process $\sigma_\beta^{(1)}$ is a non random set which coincides with the closed subset of the real line $\left[1 - h_d^-(\beta), 1 + h_d^+(\beta)\right]$, where $h_d^\pm(\beta) = \pm 2dJ\beta + o(\beta)$.*

The proofs of these theorems rely in part on the approach of [AMSZ] and [M] and in part on the lattice gas representation of the system, which we will introduce in the next subsection. More precisely, we will restate the dynamics with rates of the kind (5) in terms of a birth and death process on the set of the subsets of the lattice $\mathcal{P}$, which is naturally isomorphic to $\mathcal{S}$, and make use of the set up given in [GI1] and [GI2].

2.1 Lattice gas setting

Let $\mathcal{P}$ be the collection of the subsets of the lattice. We denote by $\mathbb{L}(\mathcal{P})$ the subspace of cylinder functions on $\mathcal{P}$ generated by linear combinations of the indicator functions of finite subsets of the lattice

$$
\begin{gathered}
\mathbb{L}(\mathcal{P}) \ni \varphi = \sum_{\alpha \subset \mathbb{Z}^d : |\alpha| < \infty} \varphi_\alpha \delta_\alpha \\
\forall \alpha \in \mathcal{P}, \quad \mathcal{P} \ni \eta \longmapsto \delta_\alpha(\eta) = \delta_{a,\eta} \in \{0,1\} \ ,
\end{gathered}
$$

where the coefficients φ_α are real numbers.

For any realizazion of $\omega \in \Omega$, the generic matrix element of the generator of the process, which in this representation we denote by $\bar{L}(\beta, \omega)$, acting on $\mathbb{L}(\mathcal{P})$ writes

$$
\left(\bar{L}(\beta,\omega)\,\delta_\alpha\right)_\eta = \sum_{x \in \mathbb{Z}^d} \left[w^{\beta,\omega}(\alpha, \alpha \triangle \{x\})\,\delta_{\eta,\alpha} - w^{\beta,\omega}(\alpha \triangle \{x\}, \alpha)\,\delta_{\eta,\alpha\triangle\{x\}} \right] \ ,
$$

where $\forall \alpha, \gamma \in \mathcal{P}$, $\alpha \triangle \gamma = (\alpha \cup \gamma) \setminus (\alpha \cap \gamma)$.

With an abuse of notation we indicate by $w^{\beta,\omega}(\alpha, \alpha\triangle\{x\})$ the transition rate from the state α to the state $\alpha\triangle\{x\}$.

Now

$$
\begin{aligned}
\bar{L}(\beta,\omega)\,\delta_\alpha &= \sum_{\eta \in \mathcal{P} : |\eta| < \infty} \left(\bar{L}(\beta,\omega)\,\delta_\alpha\right)_\eta \delta_\eta \\
&= \sum_{x \in \mathbb{Z}^d} \left[w^{\beta,\omega}(\alpha, \alpha\triangle\{x\})\,\delta_\alpha - w^{\beta,\omega}(\alpha\triangle\{x\}, \alpha)\,\delta_{\alpha\triangle\{x\}} \right] \ .
\end{aligned}
$$

Then, $\forall \varphi \in \mathbb{L}(\mathcal{P})$, we have

$$
\begin{aligned}
\bar{L}(\beta,\omega)\,\varphi &= \sum_{\alpha \in \mathcal{P} : |\alpha| < \infty} \sum_{x \in \mathbb{Z}^d} \varphi_\alpha \left[w^{\beta,\omega}(\alpha, \alpha\triangle\{x\})\,\delta_\alpha - w^{\beta,\omega}(\alpha\triangle\{x\}, \alpha)\,\delta_{\alpha\triangle\{x\}} \right] \qquad (8) \\
&= \sum_{\alpha \in \mathcal{P} : |\alpha| < \infty} \sum_{x \in \mathbb{Z}^d} w^{\beta,\omega}(\alpha, \alpha\triangle\{x\}) \left(\varphi_a - \varphi_{a\triangle\{x\}}\right) \delta_\alpha \ .
\end{aligned}
$$

This form of the generator may seem unusual at first glance, here we prove its equivalence to the classical form of generators of birth and death processes on $\mathcal{P}$.

2.1.1 Some remarks on birth and death processes for lattice gases

Notice that, for any $\omega \in \Omega$, (8) takes the form

$$\bar{L}(\beta,\omega)\varphi = \sum_{\alpha\in\mathcal{P}\,:\,|\alpha|<\infty} \left(\bar{L}(\beta,\omega)\varphi\right)_{\alpha}\delta_{\alpha} \tag{9}$$

$$\begin{aligned}\left(\bar{L}(\beta,\omega)\varphi\right)_{\alpha} &:= \sum_{x\in\alpha} w^{\beta,\omega}(\alpha,\alpha\backslash\{x\})\left(\varphi_{\alpha}-\varphi_{\alpha\backslash\{x\}}\right)+ \\ &\quad+\sum_{x\in\alpha^{c}} w^{\beta,\omega}(\alpha,\alpha\cup\{x\})\left(\varphi_{\alpha}-\varphi_{\alpha\cup\{x\}}\right) \quad \alpha\in\mathcal{P}:|\alpha|<\infty\end{aligned} \tag{10}$$

while usually the action of the generator af a birth and death process L on $\mathbb{L}(\mathcal{P})$ takes the form (9), with $(L\varphi)_{\alpha}$, $\forall\alpha\in\mathcal{P}:|\alpha|<\infty$, which can be expressed in the following two representations:

$$\left(L^{(-)}\varphi\right)_{\alpha} := \sum_{x\in\alpha}\left[w(\alpha\backslash\{x\},\alpha)\left(\varphi_{\alpha\backslash\{x\}}-\varphi_{\alpha}\right)+w(\alpha,\alpha\backslash\{x\})\left(\varphi_{\alpha}-\varphi_{\alpha\backslash\{x\}}\right)\right] \tag{11}$$

$$\left(L^{(+)}\varphi\right)_{\alpha} := \sum_{x\in\alpha^{c}}\left[w(\alpha,\alpha\cup\{x\})\left(\varphi_{\alpha}-\varphi_{\alpha\cup\{x\}}\right)+w(\alpha\cup\{x\},\alpha)\left(\varphi_{\alpha\cup\{x\}}-\varphi_{\alpha}\right)\right]. \tag{12}$$

These expressions for $(L\varphi)_{\alpha}$ are mutually equivalent and equivalent to (10). In fact, given the involution of $\mathcal{P}$

$$\mathcal{P}\ni\alpha\longmapsto\alpha^{c}=\mathbb{Z}^{d}\backslash\alpha\in\mathcal{P}\quad\alpha\subset\mathbb{Z}^{d}, \tag{13}$$

we can define the family of operators $\{\iota_{\Lambda}\}_{\Lambda\in\mathcal{P}:|\Lambda|<\infty}$ on $\mathbb{L}(\mathcal{P})$, such that

$$\begin{aligned}&\mathbb{L}(\mathcal{P})\ni\varphi\longmapsto\phi=\iota_{\Lambda}\varphi\in\mathbb{L}(\mathcal{P})\\ &\iota_{\Lambda}\delta_{\alpha}=\delta_{\alpha\triangle\Lambda}\quad\alpha\in\mathcal{P}:|\alpha|<\infty\\ &\iota_{\Lambda}\varphi=\sum_{\alpha\in\mathcal{P}\,:\,|\alpha|<\infty}\varphi_{\alpha}\delta_{\alpha\triangle\Lambda}=\sum_{\alpha\in\mathcal{P}\,:\,|\alpha|<\infty}\varphi_{\alpha\triangle\Lambda}\delta_{\alpha}\,,\quad\varphi_{\alpha\triangle\Lambda}=\varphi_{(\alpha^{c}\cap\Lambda)\cup(\alpha\cap\Lambda^{c})}\\ &\iota_{\Lambda}(\iota_{\Lambda}\varphi)=\varphi\quad\varphi\in\mathbb{L}(\mathcal{P}),\,\Lambda\in\mathcal{P}:|\Lambda|<\infty\,.\end{aligned}$$

Defining B to be the generator of a pure birth process with rates

$$w(\alpha\backslash\{x\},\alpha)\mathbf{1}_{\alpha}(x)+w(\alpha,\alpha\cup\{x\})(1-\mathbf{1}_{\alpha}(x))$$

and D the generator of a pure death process with rates

$$w(\alpha,\alpha\backslash\{x\})\mathbf{1}_{\alpha}(x)+w(\alpha\cup\{x\},\alpha)(1-\mathbf{1}_{\alpha}(x))\,,$$

we may rewrite (11) and (12) in the form

$$\left(L^{(\pm)}\varphi\right)_\alpha = \left(B^{(\pm)}\varphi\right)_\alpha + \left(D^{(\pm)}\varphi\right)_\alpha \ ,$$

where the definition of $B^{(\pm)}$ and $D^{(\pm)}$, is readily understood. Since

$$\begin{aligned} w\left(\alpha\backslash\{x\},\alpha\right) &= w\left(\alpha^c \cup \{x\},\alpha^c\right) \\ w\left(\alpha,\alpha\backslash\{x\}\right) &= w\left(\alpha^c,\alpha^c\cup\{x\}\right) \ , \end{aligned} \tag{14}$$

considering for example (11), for any finite $\Lambda \subset \mathbb{Z}^d$ we have

$$\begin{aligned} \left(\iota_\Lambda B^{(-)}\iota_\Lambda\varphi\right)_\alpha &= \sum_{x\in\alpha\triangle\Lambda} w\left(\left(\alpha\triangle\Lambda\right)\setminus\{x\},\alpha\triangle\Lambda\right)\left(\left(\iota_\Lambda\varphi\right)_{\left(\left(\alpha\triangle\Lambda\right)\setminus\{x\}\right)} - \left(\iota_\Lambda\varphi\right)_{\left(\alpha\triangle\Lambda\right)}\right) \\ &= \sum_{x\in\alpha\triangle\Lambda} w\left(\left(\alpha\triangle\Lambda\right)\setminus\{x\},\alpha\triangle\Lambda\right)\left(\varphi_{\left(\left(\alpha\triangle\Lambda\right)\setminus\{x\}\right)\triangle\Lambda} - \varphi_\alpha\right) \end{aligned}$$

and choosing $\Lambda \supset \alpha$, by (14) we get

$$\begin{aligned} \left(\iota_\Lambda B^{(-)}\iota_\Lambda\varphi\right)_\alpha &= \sum_{x\in\alpha^c\cap\Lambda} w\left(\left(\alpha^c\cap\Lambda\right)\setminus\{x\},\alpha^c\cap\Lambda\right)\left(\varphi_{\left(\left(\alpha^c\cap\Lambda\right)\setminus\{x\}\right)^c\cap\Lambda} - \varphi_\alpha\right) \\ &= \sum_{x\in\alpha^c\cap\Lambda} w\left(\left(\alpha\cup\{x\}\right)^c\cap\Lambda,\alpha^c\cap\Lambda\right)\left(\varphi_{\alpha\cup\{x\}} - \varphi_\alpha\right) \\ &= \sum_{x\in\alpha^c\cap\Lambda} w\left(\alpha\cup\{x\}\cup\Lambda^c,\alpha\cup\Lambda^c\right)\left(\varphi_{\alpha\cup\{x\}} - \varphi_\alpha\right) \\ &= \sum_{x\in\left(\alpha\cup\Lambda^c\right)^c} w\left(\left(\alpha\cup\Lambda^c\right)\cup\{x\},\alpha\cup\Lambda^c\right)\left(\varphi_{\alpha\cup\{x\}} - \varphi_\alpha\right) \ . \end{aligned}$$

We now assume the system to be confined in a box Λ with boundary conditions η. Let $\mathcal{P}_\Lambda$ be the set of the subsets of Λ. We can inject $\mathbb{L}\left(\mathcal{P}_\Lambda\right)$ in $\mathbb{L}\left(\mathcal{P}\right)$ and consider a naturally defined ι_Λ^η.

$$\mathbb{L}\left(\mathcal{P}_\Lambda\right) \ni \varphi \longmapsto \phi^\eta = \iota_\Lambda^\eta\varphi = \iota_\Lambda\left(\varphi\delta_\eta\right) \in \mathbb{L}\left(\mathcal{P}\right) \tag{15}$$

$$\iota_\Lambda^\eta\delta_\alpha = \delta_{(\alpha\cup\eta)\triangle\Lambda} = \delta_{\Lambda\backslash\alpha\cup\eta} \quad \alpha\subseteq\Lambda\,. \tag{16}$$

Independently of the choice of the boundary conditions η, $\forall\alpha\subseteq\Lambda$

$$\left(\iota_\Lambda^\eta B_{\Lambda,\eta}^{(-)}\iota_\Lambda^\eta\varphi\right)_\alpha = \sum_{x\in\Lambda\backslash\alpha} w\left(\alpha\cup\{x\},\alpha\right)\left(\varphi_{\alpha\cup\{x\}} - \varphi_\alpha\right) = \left(D_{\Lambda,\eta}^{(+)}\varphi\right)_\alpha \qquad \eta\in\mathcal{P}_{\Lambda^c},\ |\eta|<\infty\,,$$

where $B_{\Lambda,\eta}^{(\pm)}$ and $D_{\Lambda,\eta}^{(\pm)}$ denote the natural restrictions of $B^{(\pm)}$ and $D^{(\pm)}$, to $\mathbb{L}\left(\mathcal{P}_\Lambda\right)$.

To keep notation light from now on we will omit to indicate the boundary conditions where there is no danger of ambiguity. Since $\iota_\Lambda D_\Lambda^{(+)} \iota_\Lambda = B_\Lambda^{(-)}$

$$L_\Lambda^{(-)} = B_\Lambda^{(-)} + D_\Lambda^{(-)} = \iota_\Lambda \left(B_\Lambda^{(+)} + D_\Lambda^{(+)} \right) \iota_\Lambda = \iota_\Lambda L^{(+)} \iota_\Lambda$$

and $\forall \omega \in \Omega$, $\alpha \subseteq \Lambda$ (10) takes the form

$$\begin{aligned} \left(\bar{L}_\Lambda (\beta, \omega, \eta) \varphi \right)_\alpha &= \left(D_\Lambda^{(-)} (\omega, \beta, \eta) \varphi \right)_\alpha + \left(B_\Lambda^{(+)} (\omega, \beta, \eta) \varphi \right)_\alpha \\ &= \left(D_\Lambda^{(-)} (\omega, \beta, \eta) \varphi \right)_\alpha + \left(\iota_\Lambda D_\Lambda^{(-)} (\omega, \beta, \eta) \iota_\Lambda \varphi \right)_\alpha . \end{aligned} \tag{17}$$

It is worth to notice that, for any realization of the potential, $\bar{L}_\Lambda (\beta, \omega, \eta)$ commutes with ι_Λ.

3 Proof of the Theorems

In [GI1, GI2] we analysed the stochastic dynamics of a system with a ferromagnetic potential constant on $\mathbb{B}_d$, confined in a finite subset Λ of the lattice and subject to free or periodic b.c.. Making use of a formalism borrowed from quantum mechanics, we were able to represent the restriction of (2) to $\mathcal{S}_\Lambda$, in terms of a selfadjoint operator on $\mathcal{H}_\Lambda := l^2 (\mathcal{P}_\Lambda)$ unitarily equivalent to a generator of birth and death process on $\mathcal{P}_\Lambda$ of the kind (9).

Let us consider the heat bath case. Given a finite portion of the lattice Λ and a realization of the potential ω, assuming for example periodic b.c., the restriction of the generator of the process given in (8) to $\mathcal{P}_\Lambda$, takes the form (9), (10), where

$$w^{\beta,\omega} (\alpha, \alpha \triangle \{x\}) = \psi_{hb} (\beta \Delta_x H_\alpha^\omega) = \frac{1}{1 + e^{-\beta \Delta_x H_\alpha^\omega}}$$

and

$$\begin{aligned} \Delta_x H_\alpha^\omega &= H_\alpha (\omega) - H_{\alpha \triangle \{x\}} (\omega) \\ &= \mathbf{1}_\alpha (x) \left[H_\alpha (\omega) - H_{\alpha \backslash \{x\}} (\omega) \right] + \mathbf{1}_{\alpha^c} (x) \left[H_\alpha (\omega) - H_{\alpha \cup \{x\}} (\omega) \right] \end{aligned} \tag{18}$$

representing respectively (7) and (3) in the lattice gas framework. Replacing φ by δ_η for a fixed $\eta \subseteq \Lambda$ in (10), we get the generic matrix element of (9) and then of (8). We can then transform (8) into a selfadjoint operator $\bar{L}_\Lambda^s (\beta, \omega)$ on $\mathcal{H}_\Lambda$, through the unitary mapping from $\mathcal{H}_\Lambda (\beta, \omega) := l^2 \left(\mathcal{P}_\Lambda, \mu_\Lambda^{\beta,\omega} \right)$ (which is isomorphic to the restriction of $\mathcal{L} (\beta, \omega)$ to Λ) to $\mathcal{H}_\Lambda$ given by the multiplication of the elements of $\mathcal{H}_\Lambda (\beta, \omega)$ by $\sqrt{\frac{\mu_\Lambda^{\beta,\omega}}{\mu_\Lambda}}$.

Following [GI1], since $\mathcal{H}_\Lambda \cong span\{|\alpha\rangle : \alpha \subseteq \Lambda\} \cong \bigoplus_{n=0}^{|\Lambda|} \mathcal{H}_\Lambda^{(n)}$, with $\mathcal{H}_\Lambda^{(0)} \equiv \mathbb{R}$ and $\mathcal{H}_\Lambda^{(n)} := \{|\alpha\rangle \in \mathcal{H}_\Lambda : |\alpha| = n\}$, we denote by U_Λ the unitary operator

$$\begin{aligned} U_\Lambda &: \mathcal{H}_\Lambda \longrightarrow \mathcal{H}_\Lambda \\ U_\Lambda |\alpha\rangle &= \frac{1}{2^{\frac{|\Lambda|}{2}}} \sum_{\gamma \subseteq \Lambda} (-1)^{|\alpha \cap \gamma|} |\gamma\rangle \quad \alpha \subseteq \Lambda , \end{aligned} \tag{19}$$

and by E_Λ, the representation of the involution ι_Λ introduced in (15) as an operator on $\mathcal{H}_\Lambda$, that is

$$\begin{aligned} E_\Lambda : \mathcal{H}_\Lambda &\longrightarrow \mathcal{H}_\Lambda \\ E_\Lambda \left|\alpha\right\rangle &= \left|\Lambda\backslash\alpha\right\rangle \quad \alpha \subseteq \Lambda\,. \end{aligned} \tag{20}$$

Now, for any $\alpha \subseteq \Lambda$, $E_\Lambda \frac{|\alpha\rangle \pm |\alpha^c\rangle}{2} = \pm\frac{|\alpha\rangle \pm |\alpha^c\rangle}{2}$, hence $\mathcal{H}_\Lambda = \mathcal{H}_\Lambda^+ \oplus \mathcal{H}_\Lambda^-$, where $\mathcal{H}_\Lambda^\pm := span\{\frac{|\alpha\rangle \pm |\alpha^c\rangle}{2} : \alpha \subseteq \Lambda\}$. Moreover, setting

$$\bar{E}_\Lambda := U_\Lambda E_\Lambda U_\Lambda : \mathcal{H}_\Lambda \longrightarrow \mathcal{H}_\Lambda\,, \tag{21}$$

since

$$\delta_{\alpha,\gamma} = \langle \alpha | \gamma \rangle = \langle \alpha | \, U_\Lambda U_\Lambda \left|\gamma\right\rangle = 2^{-|\Lambda|} \sum_{\eta \subseteq \Lambda} (-1)^{|\alpha\cap\eta| + |\gamma\cap\eta|} = 2^{-|\Lambda|} \sum_{\eta \subseteq \Lambda} (-1)^{|(\alpha\triangle\gamma)\cap\eta|}\ ,$$

then, for any $|\alpha\rangle \in \mathcal{H}_\Lambda$,

$$\bar{E}_\Lambda \left|\alpha\right\rangle = \sum_{\gamma,\eta\subseteq\Lambda} \frac{(-1)^{|\alpha\cap\gamma| + |(\Lambda\backslash\gamma)\cap\eta|}}{2^{|\Lambda|}} \left|\eta\right\rangle = \sum_{\gamma,\eta\subseteq\Lambda} \frac{(-1)^{|\alpha\cap\gamma| - |\gamma\cap\eta| + |\eta|}}{2^{|\Lambda|}} \left|\eta\right\rangle = (-1)^{|\alpha|} \left|\alpha\right\rangle$$

so that $\mathcal{H}_\Lambda$ can also be decomposed as the direct sum of $\overline{\mathcal{H}}_\Lambda^+ := \bigoplus_{n\geq 0\,:\,2n\in\{0,..,|\Lambda|\}} \mathcal{H}_\Lambda^{(2n)}$ and $\overline{\mathcal{H}}_\Lambda^- := \bigoplus_{n\geq 0\,:\,2n+1\in\{0,..,|\Lambda|\}} \mathcal{H}_\Lambda^{(2n+1)}$. Clearly $U_\Lambda \mathcal{H}_\Lambda^\pm = \overline{\mathcal{H}}_\Lambda^\pm$.

If, for any $x \in \Lambda$, ℓ_x^Λ, $\ell_x^{\Lambda,\perp}$ denote the mutually orthogonal projectors on $\mathcal{H}_\Lambda$ such that

$$\ell_x^{\Lambda,\perp} := I_\Lambda - \ell_x^\Lambda\,;\ \ell_x^\Lambda \left|\alpha\right\rangle = \mathbf{1}_\alpha(x) \left|\alpha\right\rangle \quad \alpha \subseteq \Lambda\,. \tag{22}$$

We have

$$\begin{gathered} \bar{\ell}_x^\Lambda = U_\Lambda \ell_x^\Lambda U_\Lambda; \quad \bar{\ell}_x^{\Lambda,\perp} = U_\Lambda \ell_x^{\Lambda,\perp} U_\Lambda, \\ \ell_x^\Lambda = E_\Lambda \ell_x^{\Lambda,\perp} E_\Lambda; \quad \ell_x^{\Lambda,\perp} = E_\Lambda \ell_x^\Lambda E_\Lambda, \\ [E_\Lambda, \bar{\ell}_x^\Lambda] = [\bar{E}_\Lambda, \ell_x^\Lambda] = 0. \end{gathered} \tag{23}$$

We also denote by

$$\frac{e^{-\frac{\beta}{2}H_\Lambda^\omega}}{Z_\Lambda^{\frac{1}{2}}(\beta,\omega)} : \mathcal{H}_\Lambda(\beta,\omega) \longrightarrow \mathcal{H}_\Lambda$$

the matrix representation of the multiplication operator by $\sqrt{\frac{\mu_\Lambda^{\beta,\omega}}{\mu_\Lambda}}$.

In [GI1, GI2], comparing the Dirichlet forms, we also proved that $\bar{L}_\Lambda^s(\beta,\omega)$ admits the representation

$$\tilde{L}_\Lambda^s(\beta,\omega) = \sum_{x\in\Lambda} \tilde{L}_{x,\Lambda}^s(\beta,\omega)\ ,$$

whose matrix elements, by the definition of $\Delta_x H_\alpha^\omega$, for any two vectors $|\alpha\rangle, |\gamma\rangle$ of the base of $\mathcal{H}_\Lambda$ are

$$\langle\gamma| \tilde{L}_\Lambda^s(\beta,\omega)|\alpha\rangle = \sum_{x\in\Lambda} \langle\gamma| \tilde{L}_{x,\Lambda}^s(\beta,\omega)|\alpha\rangle \tag{24}$$

$$\begin{aligned}\langle\gamma| \tilde{L}_{x,\Lambda}^s(\beta,\omega)|\alpha\rangle &= \langle\gamma| \left\{ \mathbf{1}_\alpha(x) \frac{1}{\cosh\frac{\beta}{2}\Delta_x H_\alpha^\omega} \left[\bar{\ell}_x^\Lambda + I_\Lambda \frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega} - 1}{2} \right] + \right. \\ &\left. + \mathbf{1}_{\alpha^c}(x) \frac{1}{\cosh\frac{\beta}{2}\Delta_x H_\alpha^\omega} \left[\bar{\ell}_x^\Lambda + I_\Lambda \frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega} - 1}{2} \right] \right\} |\alpha\rangle \\ &= \langle\gamma| \left\{ \mathbf{1}_\alpha(x) \frac{1}{\cosh\frac{\beta}{2}\Delta_x H_\alpha^\omega} \left[\frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega} + 1}{2} \bar{\ell}_x^\Lambda + \frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega} - 1}{2} \bar{\ell}_x^{\Lambda,\perp} \right] + \right. \\ &\left. + \mathbf{1}_{\alpha^c}(x) \frac{1}{\cosh\frac{\beta}{2}\Delta_x H_\alpha^\omega} \left[\frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega} + 1}{2} \bar{\ell}_x^\Lambda + \frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega} - 1}{2} \bar{\ell}_x^{\Lambda,\perp} \right] \right\} |\alpha\rangle \ .\end{aligned}$$

3.1 Proof of Theorem 1

Let us set

$$L_\Lambda := \sum_{x\in\Lambda} \ell_x \,; \quad \bar{L}_\Lambda = U_\Lambda L U_\Lambda = \sum_{x\in\Lambda} \bar{\ell}_x \,,$$
$$L_\Lambda |\alpha\rangle = |\alpha| \, |\alpha\rangle \quad \alpha \subseteq \Lambda \,.$$

Lemma 4 *For any $|u\rangle \in \mathcal{H}_\Lambda$,*

$$\langle u| \bar{L}_\Lambda |u\rangle \leq 2 \langle u| L_\Lambda |u\rangle \tag{25}$$

Proof. We first notice that, for any $x \in \Lambda$, $a : \mathcal{P}_\Lambda \times \mathcal{P}_\Lambda \longrightarrow \mathbb{R}$,

$$\begin{aligned}\sum_{\alpha\subseteq\Lambda} a_{\alpha,\alpha\cup\{x\}} \mathbf{1}_{\alpha^c}(x) &= \sum_{\alpha,\eta\subseteq\Lambda} \mathbf{1}_{\alpha^c}(x) a_{\alpha,\eta} \delta_{\eta,\alpha\cup\{x\}} = \sum_{\alpha,\eta\subseteq\Lambda} \mathbf{1}_{\alpha^c}(x) a_{\alpha,\eta} \delta_{\eta,\alpha\cup\{x\}} \mathbf{1}_\eta(x) = \\ \sum_{\alpha,\eta\subseteq\Lambda} \mathbf{1}_{\eta^c}(x) a_{\eta,\alpha} \delta_{\eta\cup\{x\},\alpha} \mathbf{1}_\alpha(x) &= \sum_{\alpha,\eta\subseteq\Lambda} \mathbf{1}_{\eta^c}(x) a_{\eta,\alpha} \delta_{\eta,\alpha\setminus\{x\}} \mathbf{1}_\alpha(x) = \sum_{\alpha\subseteq\Lambda} \mathbf{1}_\alpha(x) a_{\alpha\setminus\{x\},\alpha} \,.\end{aligned} \tag{26}$$

Then, for any $|u\rangle \in \mathcal{H}_\Lambda^\pm$, we get

$$\begin{aligned}\langle u| \bar{L}_\Lambda |u\rangle &= \sum_{x\in\Lambda}\sum_{\alpha\subseteq\Lambda} \frac{u_\alpha^2 - u_\alpha u_{\alpha\triangle\{x\}}}{2} = \sum_{x\in\Lambda}\sum_{\alpha\subseteq\Lambda} \left(\frac{u_\alpha - u_{\alpha\triangle\{x\}}}{2} \right)^2 \\ &\leq \sum_{x\in\Lambda}\sum_{\alpha\subseteq\Lambda} \frac{u_\alpha^2 + u_{\alpha\triangle\{x\}}^2}{2} = \sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha} \frac{u_\alpha^2 + u_{\alpha\setminus\{x\}}^2}{2} + \sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha^c} \frac{u_\alpha^2 + u_{\alpha\cup\{x\}}^2}{2} \\ &= \sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha} u_\alpha^2 + \sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha^c} u_\alpha^2 = \sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha} u_\alpha^2 + \sum_{\alpha^c\subseteq\Lambda}\sum_{x\in\alpha^c} u_{\alpha^c}^2 = 2 \langle u| L_\Lambda |u\rangle \ ,\end{aligned}$$

but by (23) $\bar{L}_\Lambda$ commutes with E_Λ and $\mathcal{H}_\Lambda = \mathcal{H}_\Lambda^+ \oplus \mathcal{H}_\Lambda^-$. ■

Remark 2. From (1) it follows that, for any $\omega \in \Omega$, H_α^ω depends on α only through the subset of $\mathbb{B}_\Lambda$, $\partial\alpha := \{b \in \mathbb{B}_\Lambda : |b \cap \alpha| = 1\}$ then, because $\partial\alpha = \partial\alpha^c$, by (17), (20) and (23), for any realization of the potential $H_\alpha(\omega) = H_{\alpha^c}(\omega)$. Hence, for any $\beta \geq 0$, $\omega \in \Omega$, $\tilde{L}_\Lambda^s(\beta,\omega)$ commutes with E_Λ and $|g_\Lambda(\beta,\omega)\rangle := \sum_{\alpha\subseteq\Lambda} g_\alpha^\Lambda(\beta,\omega)|\alpha\rangle$, where $g_\alpha^\Lambda(\beta,\omega) := \frac{e^{-\frac{\beta}{2}H_\alpha(\omega)}}{Z_\Lambda^{(d)}(\beta,\omega)}$, which is the ground state of $\tilde{L}_\Lambda^s(\beta,\omega)$, belongs to $\mathcal{H}_\Lambda^+$.

Let β and ω be fixed. For any vector $|u\rangle \in \mathcal{H}_\Lambda$, by (24) and (26), the Dirichlet form associated to $\tilde{L}_\Lambda^s(\beta,\omega)$ can be written in the following way

$$\begin{aligned}
\langle u|\tilde{L}_\Lambda^s(\beta,\omega)|u\rangle &= \sum_{\alpha\subseteq\Lambda}\sum_{x\in\Lambda}\frac{1}{\cosh\frac{\beta}{2}\Delta_x H_\alpha^\omega}\left[\left(\frac{u_\alpha-u_{\alpha\triangle\{x\}}}{2}\right)^2+u_\alpha^2\frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega}-1}{2}\right] \qquad (27)\\
&= \sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha}\frac{1}{\cosh\frac{\beta}{2}\Delta_x H_\alpha^\omega}\left[\left(\frac{u_\alpha-u_{\alpha\backslash\{x\}}}{2}\right)^2+u_\alpha^2\frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega}-1}{2}\right]+\\
&+\sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha^c}\frac{1}{\cosh\frac{\beta}{2}\Delta_x H_\alpha^\omega}\left[\left(\frac{u_\alpha-u_{\alpha\cup\{x\}}}{2}\right)^2+u_\alpha^2\frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega}-1}{2}\right]\\
&= \sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha}\frac{1}{\cosh\frac{\beta}{2}\Delta_x H_\alpha^\omega}\left[\frac{\left(u_\alpha-u_{\alpha\backslash\{x\}}\right)^2}{2}+u_\alpha^2\frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega}-1}{2}+u_{\alpha\backslash\{x\}}^2\frac{e^{-\frac{\beta}{2}\Delta_x H_\alpha^\omega}-1}{2}\right].
\end{aligned}$$

Clearly, $\forall\omega \in \Omega$, $\tilde{L}_\Lambda^s(\beta = 0,\omega) = \bar{L}_\Lambda$.

Proposition 5 *Let $\beta \geq 0$ and $\omega \in \Omega$ be fixed. For any $|v\rangle \in \mathcal{H}_\Lambda$,*

$$\langle v|U_\Lambda\tilde{L}_\Lambda^s(\beta,\omega)U_\Lambda|v\rangle \leq (1+2b_{dJ}(\beta))\langle v|L_\Lambda|v\rangle \qquad (28)$$

where $b_{dJ}(\beta)$ is an analytic function of β such that $b_{dJ}(\beta) = c_{dJ}\beta + o(\beta)$.

Proof. Since by the previous remark it follows that $U_\Lambda\tilde{L}_\Lambda^s(\beta,\omega)U_\Lambda$ commutes with $\bar{E}_\Lambda$, we can restrict ourselves to vectors in $\overline{\mathcal{H}}_\Lambda^\pm$. Let us set $|v\rangle \in \overline{\mathcal{H}}_\Lambda^\pm$, then $U_\Lambda|v\rangle = |u\rangle \in \mathcal{H}_\Lambda^\pm$. By (27) it follows that

$$\begin{aligned}
&\frac{1}{2}\sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha^c}\frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega}-1}{\cosh\frac{\beta}{2}\Delta_x H_\alpha^\omega}u_\alpha^2 = \frac{1}{2}\sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha^c}\frac{e^{\frac{\beta}{2}\Delta_x H_{\alpha^c}^\omega}-1}{\cosh\frac{\beta}{2}\Delta_x H_{\alpha^c}^\omega}u_{\alpha^c}^2 =\\
&= \frac{1}{2}\sum_{\alpha^c\subseteq\Lambda}\sum_{x\in\alpha^c}\frac{e^{\frac{\beta}{2}\Delta_x H_{\alpha^c}^\omega}-1}{\cosh\frac{\beta}{2}\Delta_x H_{\alpha^c}^\omega}u_{\alpha^c}^2 = \frac{1}{2}\sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha}\frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega}-1}{\cosh\frac{\beta}{2}\Delta_x H_\alpha^\omega}u_\alpha^2.
\end{aligned}$$

Thus

$$\langle u|\tilde{L}_\Lambda^s(\beta,\omega)|u\rangle = \sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha}\frac{1}{\cosh\frac{\beta}{2}\Delta_x H_\alpha^\omega}\left[\frac{\left(u_\alpha-u_{\alpha\backslash\{x\}}\right)^2}{2}+u_\alpha^2\left(e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega}-1\right)\right].$$

Therefore, by (25),

$$\begin{aligned}\langle v|\, U_\Lambda \tilde{L}^s_\Lambda(\beta,\omega)\, U_\Lambda\, |v\rangle &= \langle u|\, \tilde{L}^s_\Lambda(\beta,\omega)\, |u\rangle \leq \langle v|\, L_\Lambda\, |v\rangle + b_J(\beta)\,\langle u|\, L_\Lambda\, |u\rangle \\ &= \langle v|\, L_\Lambda\, |v\rangle + b_J(\beta)\,\langle v|\, \bar{L}_\Lambda\, |v\rangle \leq (1+2b_J(\beta))\,\langle v|\, L_\Lambda\, |v\rangle\ ,\end{aligned}$$

where $b_{dJ}(\beta) := \max_{z\in[0,2dJ]}\left[\frac{e^{\frac{\beta}{2}z}-1}{\cosh\frac{\beta}{2}z}\right]$. ■

In [GI1, GI2] we introduced a new form for the generator of stochastic Ising model with transition rates, for any realization of the potential ω and $\beta \geq 0$,

$$w^{\beta,\omega}(\alpha,\alpha\triangle\{x\}) = \frac{1+e^{\beta\Delta_x H^\omega_\alpha}}{4}$$

and whose generic matrix element as an operator acting on $\mathcal{H}_\Lambda$ is

$$\langle\gamma|\, \hat{L}^s_\Lambda(\beta,\omega)\, |\alpha\rangle = \langle\gamma| \sum_{x\in\Lambda} \hat{L}^s_x(\beta,\omega)\, |\alpha\rangle \tag{29}$$

$$\begin{aligned}\langle\gamma|\, \hat{L}^s_x(\beta,\omega)\, |\alpha\rangle &= \langle\gamma| \left\{ \mathbf{1}_\alpha(x)\cosh\frac{\beta}{2}\Delta_x H^\omega_\alpha \left[\bar{\ell}^\Lambda_x + I_\Lambda \frac{e^{\frac{\beta}{2}\Delta_x H^\omega_\alpha}-1}{2}\right] + \right.\\ &\quad \left. +\mathbf{1}_{\alpha^c}(x)\cosh\frac{\beta}{2}\Delta_x H^\omega_\alpha \left[\bar{\ell}^\Lambda_x + I_\Lambda \frac{e^{\frac{\beta}{2}\Delta_x H^\omega_\alpha}-1}{2}\right]\right\} |\alpha\rangle \\ &= \langle\gamma| \left\{ \mathbf{1}_\alpha(x)\cosh\frac{\beta}{2}\Delta_x H^\omega_\alpha \left[\frac{e^{\frac{\beta}{2}\Delta_x H^\omega_\alpha}+1}{2}\bar{\ell}^\Lambda_x + \frac{e^{\frac{\beta}{2}\Delta_x H^\omega_\alpha}-1}{2}\bar{\ell}^{\Lambda,\perp}_x\right] + \right.\\ &\quad \left. +\mathbf{1}_{\alpha^c}(x)\cosh\frac{\beta}{2}\Delta_x H^\omega_\alpha \left[\frac{e^{\frac{\beta}{2}\Delta_x H^\omega_\alpha}+1}{2}\bar{\ell}^\Lambda_x + \frac{e^{\frac{\beta}{2}\Delta_x H^\omega_\alpha}-1}{2}\bar{\ell}^{\Lambda,\perp}_x\right]\right\} |\alpha\rangle\ .\end{aligned}$$

We also showed that $\hat{L}^s_\Lambda(\beta,\omega)$ admits the representation

$$\hat{L}^s_\Lambda(\beta,\omega) = \sum_{x\in\Lambda} U_\Lambda e^{\frac{\beta}{2}\mathbf{H}_\Lambda(\omega)} \ell^\Lambda_x e^{-\beta\mathbf{H}_\Lambda(\omega)} \ell^\Lambda_x e^{\frac{\beta}{2}\mathbf{H}_\Lambda(\omega)} U_\Lambda\,,$$

where

$$\begin{aligned}\mathbf{H}_\Lambda(\omega) &:= \sum_{\alpha\subseteq\Lambda} H_\alpha(\omega)\,|\alpha\rangle\langle\alpha| \simeq \mathbf{H}_\Lambda(\omega) = \sum_{b\in\mathbb{B}_\Lambda}\omega_b \mathbf{s}_b\,, \\ \mathbf{s}_b &:= \mathbf{1}_b(x)\,\mathbf{1}_b(y)\,(1-\delta_{x,y})\,\mathbf{s}_x\mathbf{s}_y\,,\end{aligned} \tag{30}$$

with $\mathbf{s}$ the Pauli matrix

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

so that, $\forall x \in \Lambda$, $\alpha \subseteq \Lambda$, $\mathbf{s}_x |\alpha\rangle = |\alpha \triangle \{x\}\rangle$ (we prefer to work in the representation where the spin flip operator is diagonal). By (24) and (29), for any two basis vectors of $\mathcal{H}_\Lambda$, $|\alpha\rangle, |\gamma\rangle$ we have

$$\langle\gamma| \tilde{L}^s_\Lambda(\beta,\omega)|\alpha\rangle \leq \langle\gamma| \hat{L}^s_\Lambda(\beta,\omega)|\alpha\rangle \ ,$$

moreover, the first order term in the expansion for small β of $\langle\gamma| \tilde{L}^s_\Lambda(\beta,\omega)|\alpha\rangle$ and $\langle\gamma| \hat{L}^s_\Lambda(\beta,\omega)|\alpha\rangle$ are equal for every Λ. Clearly, $\hat{L}^s_\Lambda(\beta,\omega)$ also commutes with E_Λ and for any $|u\rangle \in \mathcal{H}_\Lambda$, since $|u\rangle = |u^+\rangle + |u^-\rangle$, $|u^\pm\rangle \in \mathcal{H}^\pm_\Lambda$, we have

$$\begin{aligned}\langle u| \hat{L}^s_\Lambda(\beta,\omega)|u\rangle &= \sum_{\alpha\subseteq\Lambda}\sum_{x\in\Lambda}\cosh\frac{\beta}{2}\Delta_x H^\omega_\alpha\left[\left(\frac{u_\alpha - u_{\alpha\triangle\{x\}}}{2}\right)^2 + u^2_\alpha\frac{e^{\frac{\beta}{2}\Delta_x H^\omega_\alpha}-1}{2}\right] \\ &= \sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha}\cosh\frac{\beta}{2}\Delta_x H^\omega_\alpha\left[\left(\frac{u_\alpha - u_{\alpha\setminus\{x\}}}{2}\right)^2 + u^2_\alpha\frac{e^{\frac{\beta}{2}\Delta_x H^\omega_\alpha}-1}{2}\right] .\end{aligned} \tag{31}$$

Proceding as in Proposition 5, we get

$$\langle u| \hat{L}^s_\Lambda(\beta,\omega)|u\rangle \leq (1+2b'_{dJ}(\beta))\langle u|\bar{L}_\Lambda|u\rangle \ , \tag{32}$$

where $b'_{dJ}(\beta) := \max_{z\in[0,2dJ]}\left[\left(e^{\frac{\beta}{2}z}-1\right)\cosh\frac{\beta}{2}z + \cosh\frac{\beta}{2}z - 1\right]$. Comparing the Dirichlet forms of $\hat{L}^s_\Lambda(\beta,\omega)$ and $\tilde{L}^s_\Lambda(\beta,\omega)$, we proved in [GI2] this process to converge to the equilibrium state at high temperature faster than the heat-bath one.

Remark 3. The relative bounds (28) (32) are independent of Λ and extends straightforwardly to the quadratic forms associated to the operator $\tilde{L}^s(\beta,\omega)$ and $\hat{L}^s(\beta,\omega)$ acting on $\mathcal{H}$. Therefore, by standard argument of perturbation theory (see for example [K]) (28) implies the analyticity of the projectors

$$P_n(\beta,\omega) := \oint_{\{z\in\mathbb{C}\,:\,|z-n|\leq r(\beta)\}}\frac{dz}{2\pi i}\frac{1}{Iz - A(\beta,\omega)} \quad n\in\mathbb{N}\,,$$

where $A(\beta,\omega)$ is either $\tilde{L}^s(\beta,\omega)$ or $\hat{L}^s(\beta,\omega)$, for sufficiently small values of β.

3.1.1 Lower bound $g^-_d(\beta)$

By Remark 3.1, we can make use of perturbation theory and, for sufficently small values of β and any realization of the potential, we can write

$$\hat{L}^s_\Lambda(\beta,\omega) = U_\Lambda L_\Lambda U_\Lambda + \beta U_\Lambda T^{(1)}_\Lambda(\omega)U_\Lambda + \bar{T}_\Lambda(\beta,\omega) \ ,$$

where

$$T^{(1)}_\Lambda(\omega) := \frac{1}{2}\sum_{x\in\Lambda}[[\mathbf{H}_\Lambda(\omega), \ell^\Lambda_x], \ell^\Lambda_x]$$

is the first term in the expansion of $\hat{L}^s_\Lambda(\beta,\omega)$ and $\bar{T}_\Lambda(\beta,\omega)$ is such that $\langle u|\,\bar{T}_\Lambda(\beta,\omega)\,|u\rangle \leq \beta^2 C(d,J)$, with $C(d,J)$ a positive constant.

Since, by definition of U_Λ, $U_\Lambda \hat{L}^s_\Lambda(\beta,\omega)\,U_\Lambda$ and $\hat{L}^s_\Lambda(\beta,\omega)$ have the same spectrum, the eigenspace corresponding to $\xi_1(L_\Lambda)=1$ is $span\{|y\rangle : y\in\Lambda\}$ and

$$\begin{aligned}\langle z|\,T^{(1)}_\Lambda(\omega)\,|y\rangle &= \frac{1}{2}\,\langle z|\sum_{x\in\Lambda}[[\mathbf{H}_\Lambda(\omega),\ell^\Lambda_x],\ell^\Lambda_x]\,|y\rangle \\ &= \frac{1}{2}\sum_{x\in\Lambda}\langle z|\,\mathbf{H}_\Lambda(\omega)\,|y\rangle\,(\delta_{x,y}+\delta_{x,z}-2\delta_{z,x}\delta_{x,y}) \\ &= \langle z|\,\mathbf{H}_\Lambda(\omega)\,|y\rangle - \delta_{z,y}\,\langle y|\,\mathbf{H}_\Lambda(\omega)\,|y\rangle\ ,\end{aligned} \tag{33}$$

where by (30)

$$\langle z|\,\mathbf{H}_\Lambda(\omega)\,|y\rangle = \sum_{b\in\mathbb{B}_\Lambda}\omega_b\,\langle z|\,\mathbf{s}_b\,|y\rangle = \sum_{b\in\mathbb{B}_\Lambda}\omega_b\,\langle z|\{y\}\triangle b\rangle = \sum_{b\in\mathbb{B}_\Lambda}\omega_b\mathbf{1}_{\{z,y\}}(b) = \omega_{z,y}\,. \tag{34}$$

Moreover, looking at the expansion in β of the Dirichlet forms of $\tilde{L}^s_\Lambda(\beta,\omega)$ and $\hat{L}^s_\Lambda(\beta,\omega)$, we realize that these operators coincides up to first order. Hence, we get $\xi_1\left(\tilde{L}^s_\Lambda(\beta,\omega)\right) \geq g^-_d(\beta)$, with $g^-_d(\beta)$ analytic function of β such that

$$g^-_d(\beta) := 1-\beta\sup_{z\in\Lambda}\sum_{y\in\Lambda}|\omega_{x,y}| + o(\beta) = 1-2\beta dJ + O\left(\beta^2\right)\ . \tag{35}$$

Notice that all the above estimates, which are independent of Λ, hold in infinite volume as well.

Remark 4. Since the ω's are bounded, the last result implies the existence of a value of $\beta_d(J)$ smaller than the critical one $\beta_c(d,\omega)$, such that for $\mathbb{P}$ a.e. ω, if $\beta\in[0,\beta_d(J))$, the process is ergodic. Hence, by the reversibility with respect to the Gibbs measure, we get the uniqueness of the Gibbs state. Furthermore, the unique element $\mu^{\beta,\omega}$ of $\mathcal{G}(\beta,\omega)$ has the property

$$\mu^{\beta,\omega}(A) = \mu^{\beta,T_z\omega}(\tau_z A) \quad A\subset\mathcal{S},\ z\in\mathbb{Z}^d\,, \tag{36}$$

where

$$\tau_z A := \left\{\sigma\in\mathcal{S} : \forall x\in\mathbb{Z}^d \quad \sigma_x = \eta_{x-z} = (\tau_z\eta)_x\,,\ \eta\in A\right\}\ .$$

Let $\{\Theta_z\}_{z\in\mathbb{Z}^d}$ be the unitary group of operators on $\mathcal{L}(\beta,\omega)$ generated by the group $\{\tau_z\}_{z\in\mathbb{Z}^d}$ that is,

$$(\Theta_z\varphi)(\sigma) = \varphi\left(\tau_z^{-1}\sigma\right) \quad \varphi\in\mathcal{L}(\beta,\omega)\ .$$

Then, by the previous remark, we get that $\forall z\in\mathbb{Z}^d$ the Hilbert spaces $\mathcal{L}(\beta,\omega)$ and $\mathcal{L}(\beta,T_z^{-1}\omega)$ are unitary equivalent (isomorphic) by the unitary mapping Θ_z

$$\Theta_z : \mathcal{L}(\beta,\omega) \longmapsto \mathcal{L}\left(\beta,T_z^{-1}\omega\right)$$

and by the representation (2) of $L(\beta,\omega)$, we have

$$\Theta_z L(\beta,\omega)\,\Theta_z^{-1} = L\left(\beta, T_z^{-1}\omega\right) \ ,$$

which implies that, at least for $\beta \in [0, \beta_d(J))$, the family of operators and spaces $(L(\beta,\omega), \mathcal{L}(\beta,\omega))$ is a metrically transitive family with respect to the unitary group of lattice translation $\{\Theta_z\}_{z\in\mathbb{Z}^d}$. Hence, (see [PF] and [AMSZ] remark 4) the spectrum of $L(\beta,\omega)$ is non-random for $\mathbb{P}$-a.e. ω.

Remark 5. To get an upper bound for the spectral gap of the generator of the process we can compute the Dirichlet form of $L(\beta,\omega)$ with respect to the function of the empirical magnetization

$$\phi_\Lambda := \sum_{x\in\Lambda}\frac{\sigma_x}{|\Lambda|} - \mu^{\beta,\omega}\left(\sum_{x\in\Lambda}\frac{\sigma_x}{|\Lambda|}\right) .$$

We have

$$\begin{aligned}\langle\phi_\Lambda, L(\beta,\omega)\,\phi_\Lambda\rangle_{\beta,\omega} &= \frac{1}{2}\int \mu^{\beta,\omega}(d\sigma)\sum_{x\in\mathbb{Z}^d} w_x^{\beta,\omega}(\sigma)\left[\sum_{y\in\Lambda}\frac{\sigma_y}{|\Lambda|}(1-2\delta_{x,y}) - \sum_{y\in\Lambda}\frac{\sigma_y}{|\Lambda|}\right]^2 \\ &= 2\int \mu^{\beta,\omega}(d\sigma)\sum_{x\in\Lambda}\frac{w_x^{\beta,\omega}(\sigma)}{|\Lambda|^2} .\end{aligned}$$

Dividing by the $\mathcal{L}(\beta,\omega)$ norm of ϕ_Λ

$$\frac{1}{|\Lambda|^2}\sum_{x,y\in\Lambda}\left[\mu^{\beta,\omega}(\sigma_x\sigma_y) - \mu^{\beta,\omega}(\sigma_x)\,\mu^{\beta,\omega}(\sigma_y)\right]$$

we have that the spectral gap is smaller than

$$\frac{2\int \mu^{\beta,\omega}(d\sigma)\sum_{x\in\Lambda} w_x^{\beta,\omega}(\sigma)}{\sum_{x,y\in\Lambda}\left[\mu^{\beta,\omega}(\sigma_x\sigma_y) - \mu^{\beta,\omega}(\sigma_x)\,\mu^{\beta,\omega}(\sigma_y)\right]} .$$

By the ergodicity of the random field ω with respect to the lattice translations, the last expression becomes

$$\frac{2\int \mathbb{P}(d\omega)\int \mu^{\beta,\omega}(d\sigma)\, w_0^{\beta,\omega}(\sigma)}{\int \mathbb{P}(d\omega)\sum_{y\in\mathbb{Z}^d}\left[\mu^{\beta,\omega}(\sigma_0\sigma_y) - \mu^{\beta,\omega}(\sigma_0)\,\mu^{\beta,\omega}(\sigma_y)\right]} , \tag{37}$$

where

$$\chi^{d,\omega}(\beta) := \sum_{x\in\mathbb{Z}^d}\left[\mu^{\beta,\omega}(\sigma_x\sigma_0) - \mu^{\beta,\omega}(\sigma_x)\,\mu^{\beta,\omega}(\sigma_0)\right] \quad \omega\in\Omega$$

is the *susceptibility* relative to a realization of the potential. In the ferromagnetic case ($\omega_b \geq J^- > 0$, $\forall b \in \mathbb{B}_d$), by the Griffiths inequalities (see for example [L] page 186), we have

that $\chi^{d,\omega}(\beta)$ is larger than or equal to the susceptibility relative to the configuration of the potential constantly equal to J^-, $\chi^{d,J^-}(\beta)$, which is known to be a function of β diverging when β approachs its critical value $\beta_c(d,\omega)$ from below. In particular, in the two-dimensional case, $\chi^{2,J^-}(\beta)$ is proportional to $|\beta-\beta_c(2,J^-)|^{-\frac{7}{4}}$ ([H] Theorem 2.11). Then by (6), (37) is smaller than

$$\frac{2\psi(-\beta 4dJ)\vee\psi(\beta 4dJ)}{\chi^{d,J^-}(\beta)} .$$

3.1.2 Upper bound $g_d^+(\beta)$

Since, for any $\beta\geq 0$ and $\omega\in\Omega$,

$$\langle u|\tilde{L}_\Lambda^s(\beta,\omega)|u\rangle\leq\langle u|\hat{L}_\Lambda^s(\beta,\omega)|u\rangle\quad |u\rangle\in\mathcal{H}_\Lambda ,$$

from (33) and (34) we get $\xi_1\left(\tilde{L}_\Lambda^s(\beta,\omega)\right)\leq g_d^+(\beta)$, with $g_d^+(\beta)$ analytic function of β such that

$$g_d^+(\beta):=1+\beta\sup_{z\in\Lambda}\sum_{y\in\Lambda}|\omega_{x,y}|+o(\beta)=1+2\beta dJ+O\left(\beta^2\right) . \tag{38}$$

3.1.3 $\sigma_\beta^{(1)}\subseteq\left[g_d^-(\beta J),g_d^+(\beta J)\right]$

We just notice that, for β smaller than $\beta_d(J)$, we have $g_d^-(\beta)=1-2\beta Jd+O(\beta^2)$ and $g_d^+(\beta)=1+2\beta Jd+O(\beta^2)$, which implies that, for such values of β,

$$\sigma_\beta^{(1)}\subseteq[1-2\beta Jd,1+2\beta Jd] . \tag{39}$$

3.2 Proof of Theorem 2

Here we mimic the second part of the proof of Theorem 3 in [AMSZ] and consider Ω as a topological space endowed with the Schwartz topology, which we will denote by $\mathcal{D}_{\mathbb{B}_d}$. We will denote by *supp*$\mathbb{P}$ the support of $\mathbb{P}$ as a function on $\mathcal{D}_{\mathbb{B}_d}$. Let ζ be any realization of the potential constant on $\mathbb{B}_d$ which belongs to the support of $\mathbb{P}$, namely

$$\zeta=\{\omega_b=\zeta\in\mathbb{R},\forall b\in\mathbb{B}\}\in supp\mathbb{P}$$

and denote by $\mathcal{C}_{\mathbb{B}_d}\subset\mathcal{D}_{\mathbb{B}_d}$ the collection of all such realizations of the potential.

Theorem 3 of [AMSZ] uses the explicit representation of the matrix elements of the generator for the 1-d model to prove the weak continuity of the spectral measure. All is really needed is that the matrix elements of the generator and thus the semigroup are smooth functions of the potential. In higher dimension we rely on (**??**), which in particular ensures the necessary regularity.

It is proved in Theorem 2.2 of [M] that, for any constant realizations ζ of the potential in $supp\mathbb{P}$, there exists a value $\beta_d^{(1)}(\zeta) > 0$ such that, for any $|\beta| < \beta_d^{(1)}(\zeta)$, we obtain

$$[1 - a_d(\beta\zeta), 1 + a_d(\beta\zeta)] \subseteq \sigma_\beta^{(1)},$$

with

$$a_d(r) := \max_{\lambda\in\mathbb{T}^d} |a_d(\lambda, r)| \ . \tag{40}$$

Consequently, if $\beta_d^{(1)} := \inf_{\zeta\in supp\mathbb{P}} \beta_d^{(1)}(\zeta) > 0$ and $\beta_d^*(J) := \beta_d(J) \wedge \beta_d^{(1)}$, then, $\forall\beta \in [0, \beta_d^*(J))$,

$$[1 - \bar{a}_d(\beta), 1 + \bar{a}_d(\beta)] \subseteq \bigcup_{\zeta\in\mathcal{C}_{\mathbb{B}_d}} \sigma_\beta^{(1)}\left(L^{(1)}(\beta,\zeta)\right) \subseteq \sigma_\beta^{(1)} \tag{41}$$

with

$$\bar{a}_d(\beta) := \max_{\zeta\in\mathcal{C}_{\mathbb{B}_d}} a_d(\beta\zeta)$$

which is an analytic function of β. Thus $f_d^\pm(\beta) := \pm\bar{a}_d(\beta)$. Since $\bar{a}_d(\beta) = a_d(\beta\zeta)$ for ζ such that $|\zeta| = J$, then for small values of β, $\bar{a}_d(\beta) = 2dJ\beta + o(\beta)$, where the linear term in β is the same in the expansion of (35), as well as in (38).

3.3 Proof of Theorem 3

Because the family of operators and spaces $(L(\beta,\omega), \mathcal{L}(\beta,\omega))$ is metrically transitive with respect to lattice translations, $\sigma_\beta^{(1)}$ is a non random set (see remark 2). Thus, for every $\beta \in [0, \beta_d^*(J))$, at first order in β, by (39) and (41) we obtain

$$[1 - 2dJ\beta, 1 + 2dJ\beta] \subseteq \sigma_\beta^{(1)} \subseteq [1 - 2\beta dJ, 1 + 2\beta dJ] \ .$$

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References

[AMSZ] S. Albeverio, R. Minlos, E. Scacciatelli, E. Zhizhina *Spectral Analysis of the Disordered Stochastic 1-D Ising Model* Comm. Math. Phys. **204**, 651-668 (1999).

[CMM] Cesi F., Maes, C. and Martinelli F., *Relaxation in Disordered Magnets in the the Griffiths Regime* Comm. Math. Phys. **188**, 135–173 (1997).

[GI1] M. Gianfelice, M. Isopi *Quantum Methods for Interacting Particle Systems II, Glauber Dynamics for Ising spin Systems* Markov Processes and Relat. Fields **4**, 411–428 (1998).

[GI2] M. Gianfelice, M. Isopi *Erratum and Addenda to Quantum Methods for Interacting Particle Systems II, Glauber Dynamics for Ising spin Systems* Markov Processes and Relat. Fields **9**, 513–516 (2003).

[GZ] Guionnet A. and Zegarlinski B., *Decay to Equilibrium in Random Spin Systems on a Lattice* Comm. Math. Phys. **181**, 703–732 (1996).

[H] R. Holley *On the Asymptotics of the Spin-Spin Autocorrelation Function In Stochastic Ising Models Near the Critical Temperature* Progress in Probability n. **19** Spatial Stochastic Processes Birkäuser Boston 1991.

[K] T. Kato *Perturbation theory for linear operators* Springer-Verlag 1966.

[L] T. Liggett *Interacting Particle Systems* Springer-Verlag 1985.

[M] R. Minlos *Invariant Subspaces of the Stochastic Ising High Temperature Dynamics* Markov Processes Relat. Fields 2, 263–284 (1996).

[PF] L. Pastur, A. Figotin *Spectra of random and almost periodic operators* Springer-Verlag 1991.

[SZ] H. Spohn, E. Zhizhina, *Long-time behavior for the 1-D stochastic Ising model with unbounded random couplings* J. Statist. Phys. **111**, 419–431 (2003).

[Ze] Zegarlinski B., *Strong Decay to Equilibrium in one Dimensional Random Spin Systems* J. Stat. Phys. **77**, 717—732 (1994).

[Zh] E. Zhizhina, *The Lifshitz tail and relaxation to equilibrium in the one-dimensional disordered Ising model* J. Statist. Phys. **98**, 701–721 (2000).